13/1/20

ΠΑΛΙΑ $\boxed{\text{ΘΕΜΑΤΑ}}$

- Έστω η εξίσωση $\frac{\partial u}{\partial t} + u\cdot\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}$

Να βρεθούν οι m, n ώστε $u = t^m f(\eta)$, $\eta = x t^n$ να λύνει την εξίσωση

ΛΥΣΗ:

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial t} \\ \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} \end{aligned}\right]$$

$$\Leftrightarrow \quad t^m f' n x t^{n-1} + m t^{m-1} f = \frac{\partial u}{\partial t}$$

$$t^m f' t^n + 0 = \frac{\partial u}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(t^m f' t^n\right) = \frac{\partial}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}$$

$$= t^m f'' t^n \cdot t^n$$

Αντικαθιστώ:

$$n t^{m+n-1} x f' + m t^{m-1} f + t^m f\, t^{m+n} f' = t^{m+2n} f''$$

$$\underset{=n\cdot\eta f'}{}$$

$$\Leftrightarrow \quad m f + n x t^n f' + t^{m+n+1} f f' = t^{2n+1} f'' \quad \Leftrightarrow$$

είναι σταθ.
δε μας ενοχλούν

Θέλω να έχω εξίσ. μόνο με f κ. η
Να απλοποιούνται οι όροι του x

$$\Leftrightarrow \begin{cases} m + n + 1 = 0 & \Rightarrow m = -1/2 \\ 2n + 1 = 0 & \Rightarrow n = -1/2 \end{cases}$$

Τότε: $\boxed{-\frac{1}{2} f - \frac{1}{2}\eta f' + f f' = f''}$

- $f(x) = g(x) + g(-x) + \hat{g}(x) + \hat{g}(-x)$,
  όπου $\hat{g}$ ο FT της g
  Ν.δ.ο. $f(x) = \hat{f}(x)$

ΛΥΣΗ:

$$\hat{f}(x) = \int_{-\infty}^{\infty} \left(g(x) + g(-x) + \hat{g}(x) + \hat{g}(-x) \right) e^{ikx}\, dx$$

$$\Rightarrow \hat{f}(x) = \hat{g}(x) + \hat{g}(-x) + g(-x) + g(x)$$

$$\hat{g}(k) = \int_{-\infty}^{\infty} g(x) e^{ikx}\, dx$$

$$\hat{\hat{g}}(l) = \int_{-\infty}^{\infty} \hat{g}(k) e^{ikl}\, dk = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(x) e^{ikx}\, dx \right) e^{ilk} dk$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)\, e^{ik(x+l)}\, dk\, dx = \int_{-\infty}^{\infty} g(x) \underbrace{\left(\int_{-\infty}^{\infty} e^{i(l+x)k} dk \right)}_{\delta(-x)} dx$$

$$= \int_{-\infty}^{\infty} g(x)\, \delta(-x)\, dx = g(-x)$$

- $f(x): \mathbb{R} \to \mathbb{R}$

  Ν.δ.ο. η $|\hat{f}(\kappa)|^2$ είναι άρτια

  $= \hat{f}(\kappa) \cdot \overline{\hat{f}(\kappa)}$

ΛΥΣΗ:

$f(x) \in \mathbb{R}$

$$\hat{f}(\kappa) = \int_{-\infty}^{\infty} f(x) e^{i\kappa x} dx = \int_{-\infty}^{\infty} f(x) e^{-i\kappa x} dx$$

$$|\hat{f}(-\kappa)|^2 = \hat{f}(-\kappa) \cdot \overline{\hat{f}(-\kappa)} = \int_{-\infty}^{\infty} f(x) e^{-i\kappa x} dx \cdot \int_{-\infty}^{\infty} f(x) e^{i\kappa x} dx$$

$$= |\hat{f}(\kappa)|^2$$

άρα $|\hat{f}|^2$ άρτια

- Δίνεται η εξίσωση $\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} - 6u \frac{\partial u}{\partial x} = 0$

i) Αν $a + 6b = 0 \Rightarrow \begin{cases} T = t \\ X = x + at \\ U = u + b \end{cases}$

Ν.δ.ο. αφήνει την εξίσωση αναλλοίωτη

ΛΥΣΗ: (αλλαγή μεταβλητής)

Αν βρω μια λύση στο x μπορώ να βρω άλλη για το X (με την αλλαγή μεταβλ.)

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial T} \cdot \frac{\partial T}{\partial t} + \frac{\partial u}{\partial X} \cdot \frac{\partial X}{\partial t} \\ \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial X} \cdot \frac{\partial X}{\partial x} + \frac{\partial u}{\partial T} \cdot \frac{\partial T}{\partial x} \end{aligned}\right\} \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial u}{\partial t} = \frac{\partial u}{\partial T}\cdot 1 + \frac{\partial u}{\partial x}\cdot a$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}\cdot 1 + \frac{\partial u}{\partial T}\cdot 0$$

$$\Leftrightarrow \frac{\partial u}{\partial t} = \frac{\partial u}{\partial T} + \frac{\partial u}{\partial x}\cdot a$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}$$

$$\rightarrow \frac{\partial^3 u}{\partial x^3} = \frac{\partial^3 u}{\partial x^3}$$

Αντικαθιστώ:

$$\frac{\partial u}{\partial T} + \frac{\partial u}{\partial x}\cdot a + \frac{\partial^3 u}{\partial x^3} - 6u\frac{\partial u}{\partial x} = 0$$

$$\Leftrightarrow \frac{\partial u}{\partial T} + \boxed{\frac{\partial u}{\partial x}\cdot a} + \frac{\partial^3 u}{\partial x^3} - 6(u\boxed{-b)\frac{\partial u}{\partial x}} = 0$$

$a + 6b = 0$ άρα

$$\Leftrightarrow \frac{\partial u}{\partial T} + \frac{\partial^3 u}{\partial x^3} - 6u\frac{\partial u}{\partial x} = 0$$

ii) Να βρεθεί η σταθερά $u = \frac{A}{x^2}$ τ.ω. να λύνει την εξίσωση $\frac{\partial(A/T^2)}{\partial t} = 0$

ΛΥΣΗ:

$$\frac{\partial\left(\frac{A}{x^2}\right)}{\partial x} = -\frac{2A}{x^3}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{2}{x^3}A\right) = \frac{6A}{x^5}$$

$$\frac{\partial^3 u}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{6A}{x^5}\right) = -\frac{30A}{x^9}.$$

iii) Να βρείτε ακόμα μία λύση (από το α))

ΛΥΣΗ:

$$u = \frac{A}{x^2} + B = \frac{A}{(x+a+1)^2} - \frac{a}{6} \quad \forall a \in \mathbb{R}$$

είναι λύση

iv) Ν.δ.ο. για επαρκείς φθίνουσες συνθήκες στα άπειρα $\Rightarrow \frac{\partial^2 u}{\partial x^2} = 0$

ΛΥΣΗ:

$$\int_{-\infty}^{\infty} u\,dx = \text{σταθερά} \Leftrightarrow \frac{d}{dt}\int_{-\infty}^{\infty} u\,dx = 0$$

→ συνάρτηση του t

(το έχουμε λύσει)

Ολοκληρώνω την εξίσωση:

$$\int_{-\infty}^{\infty} \frac{\partial u}{\partial t}\,dx = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} 6u\frac{\partial u}{\partial x}\,dx = 3\frac{\partial u}{\partial x}$$

- $$\frac{\partial u}{\partial t} - u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^2 \partial t} = 0$$

N.Ʒ.O. : i) $\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial t} + \frac{1}{2}u^2\right) = 0$

$$-\frac{\partial^3 u}{\partial x^2 \partial t} - u\frac{\partial u}{\partial x} = 0$$

ii) $\frac{1}{2}\cdot\frac{\partial}{\partial t}(u^2 + u_x^2) - \frac{\partial}{\partial x}\left(uu_{xt} + \frac{1}{3}u^3\right) = 0$

$$\frac{1}{2}\cdot 2u\cdot\frac{\partial u}{\partial t} + \frac{1}{2}\cdot 2\cdot\frac{\partial^2 u}{\partial x \partial t} - \frac{\partial^3 u}{\partial x^2 \partial t} - \frac{\cancel{3}u^2}{\cancel{3}}\frac{\partial u}{\partial x} = 0$$